Asteroid Rendezvous Missions

An Application of Optimal Control

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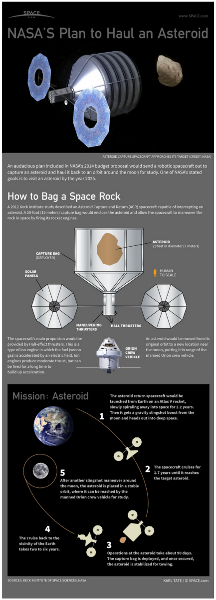
Missions to Asteroids

Launch	Mission	Time (years)	Δv (m/s)
1978	*ISEE-3*	7.08	430
1989	*Galileo*	2.03	1300
1996	*NEAR*	1.36	1450
1997	*Cassini*	2.21	500
1998	*Deep Space 1*	2.91	1300
1999	*Stardust*	5.34	230
2003	*Hayabusa*	2.32	1400
2004	*Rosetta*	4.51	2200
2005	*DIXI*	5.63	190
2007	*Dawn*	3.76	10000
2014	*Hayabusa 2*	4.00*	
2016*	*Osiris-Rex*	2.00*	
2020*	*ARM*	4.00*	



NASA'S Plan to Haul an Asteroid

ASTEROID CAPTURE SPACECRAFT APPROACHES ITS TARGET (CREDIT: NASA)

An audacious plan included in NASA's 2014 budget proposal would send a robotic spacecraft out to capture an asteroid and haul it back to an orbit around the moon for study. One of NASA's stated goals is to visit an asteroid by the year 2025.

How to Bag a Space Rock

A 2012 Keck Institute study described an Asteroid Capture and Return (ACR) spacecraft capable of intercepting an asteroid. A 50-foot (15 meters) capture bag would enclose the asteroid and allow the spacecraft to maneuver the rock in space by firing its rocket engines.

The spacecraft's main propulsion would be provided by Hall-effect thrusters. This is a type of ion engine in which the fuel (xenon gas) is accelerated by an electric field. Ion engines produce moderate thrust, but can be fired for a long time to build up acceleration.

An asteroid would be moved from its original orbit to a new location near the moon, putting it in range of the manned Orion crew vehicle.

Mission: Asteroid

1 The asteroid return spacecraft would be launched from Earth on an Atlas V rocket, slowly spiraling away into space for 2.2 years. Then it gets a gravity slingshot boost from the moon and heads out into deep space.

2 The spacecraft cruises for 1.7 years until it reaches the target asteroid.

3 Operations at the asteroid take about 90 days. The capture bag is deployed, and once secured, the asteroid is stabilized for towing.

4 The cruise back to the vicinity of the Earth takes two to six years.

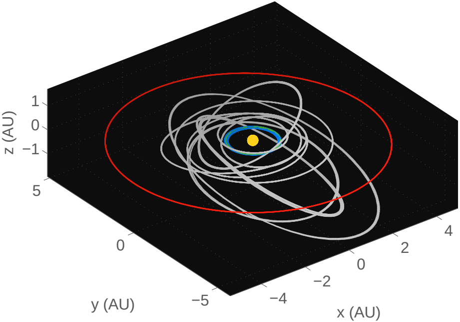
5 After another slingshot maneuver around the moon, the asteroid is placed in a stable orbit, where it can be reached by the manned Orion crew vehicle for study.

SOURCES: KECK INSTITUTE OF SPACE SCIENCES, NASA

KARL TATE / © SPACE.com

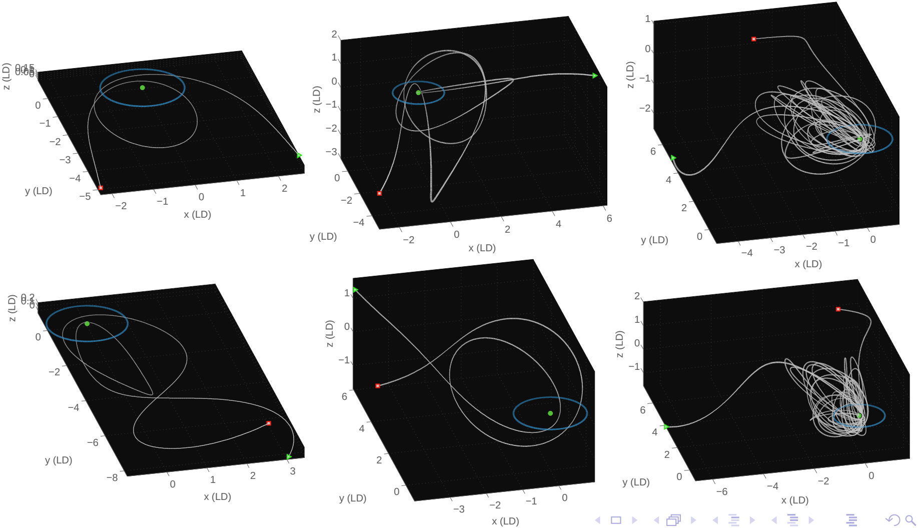
Missions to Asteroids

Asteroid	Perigee (LD)	Diameter
Ida	680.3	31 km
Mathilde	369.7	53 km
Gaspra	326.1	18 km
Borrelly	188.1	8 km
Halley	164.7	11 km
Churyumov	153.4	4 km
Braille	123.3	2 km
Eros	59.0	17 km
Itokawa	13.6	0.54 km
2006RH$_{120}$	0.7	0.003 km

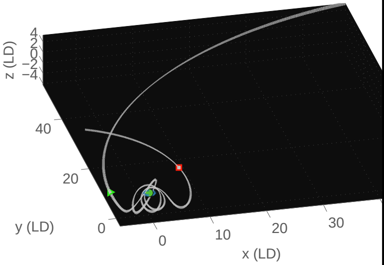


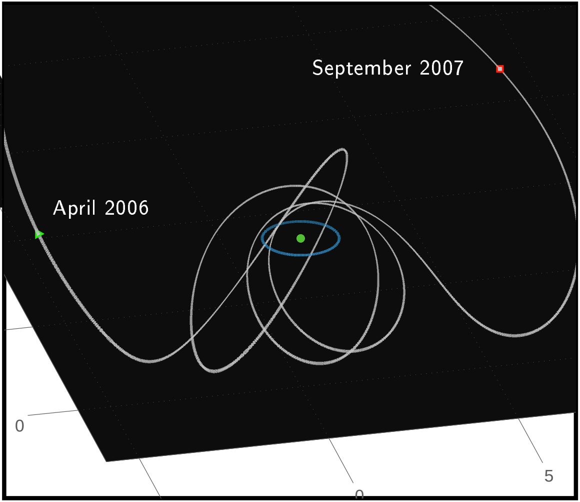
Granvik et al. [2011]

- Database of 16,923 simulated ***minimoons***
- At least one in orbit at any time (1-meter diameter)



Asteroid 2006RH$_{120}$

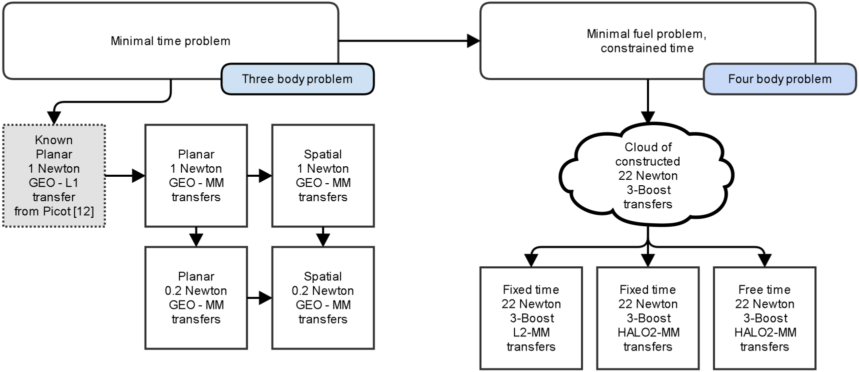




Defn. *temporary capture* :

1. the geocentric Keplerian energy $E < 0$
2. the geocentric distance is less than three Hill radii ≈ 0.03 AU.
3. at least one full revolution around Earth in co-rotating frame

Paths to solutions



Minimal time problem

Three body problem

Known
Planar
1 Newton
GEO - L1
transfer
from Picot [12]

Planar
1 Newton
GEO - MM
transfers

Spatial
1 Newton
GEO - MM
transfers

Planar
0.2 Newton
GEO - MM
transfers

Spatial
0.2 Newton
GEO - MM
transfers

Minimal fuel problem,
constrained time

Four body problem

Cloud of
constructed
22 Newton
3-Boost
transfers

Fixed time
22 Newton
3-Boost
L2-MM
transfers

Fixed time
22 Newton
3-Boost
HALO2-MM
transfers

Free time
22 Newton
3-Boost
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transfers

Modern optimal control theory : the Pontryagin's Maximum Principle

Consider a *general* optimal control problem

$$\left\{ \begin{array}{c} \dot{x}(t) = f(t, x(t), u(t)) \\ \min_{u(.)\in U} \int_{t_0}^{t_f} f_0(t, x(t), u(t))dt + g(t_f, x_f) \\ x(t_0) = x_0 \in M_0 \subset M, \; x(t_f) = x_f \in M_1 \subset M \end{array} \right.$$

In 1958, the Russian mathematician Lev Pontryagin stated a fundamental *necessary first order* optimality condition, known as the *Pontryagin's Maximum Principle*.

This result generalizes the *Euler-Lagrange* equations from the theory of calculus of variations.

Modern optimal control theory : the Pontryagin's Maximum Principle

If u is optimal on $[0, t_f]$ then there exists $p \in T_x^*M$ and $p^0 \in \mathbb{R}^-$ such that $(p^0, p) \neq (0, 0)$ and almost everywhere in $[t_0, t_f]$ there holds

- $z(t) = (x(t), p(t)) \in T^*M$ is *solution* to the *pseudo-Hamilonian system*

  $$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p^0, p(t), u(t)), \ \dot{p}(t) = -\frac{\partial H}{\partial q}(x(t), p^0, p(t), u(t))$$

  where

  $$H(x, p^0, p, u) = p^0 f^0(x, u) + < p, f(x, u) > ;$$

- *maximization condition*

  $$H(x(t), p^0, p(t), u(t)) = \max_{v \in U \subseteq N} H(x(t), p^0, p(t), v) ;$$

- *transversality condition*

  $$p(0) \perp T_{x(0)}M_0 \text{ and } p(t_f) - p^0 \frac{\partial g}{\partial t}(t_f, x(t_f)) \perp T_{x(t_f)}M_1).$$

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- A triplet (x, p, u) satisfying these 3 conditions is called an *extremal solution*.

Modern optimal control theory : the Pontryagin's Maximum Principle

Locally $U = \mathbb{R}^n \Rightarrow \frac{\partial H}{\partial u} = 0$.

Assumption : The quadratic form $\frac{\partial^2 H}{\partial u^2}$ is **negative definite** along the extremal $(x(t), p(t), u(t))$.

Implicit fonction theorem

$\Rightarrow$ In a neighborhood of u, extremal controls are *feedback controls* i.e smooth functions

$$u_r(t) = u_r(x(t), p(t))$$

$\Rightarrow$ extremal curves are pairs $(x(t), p(t))$ solutions of the *true Hamiltonian system*

$$\dot{x}(t) = \frac{\partial H_r}{\partial p}(x(t), p(t)), \;\; \dot{p}(t) = -\frac{\partial H_r}{\partial x}(x(t), p(t))$$

where H_r is the *true Hamiltonian function* defined by

$$H_r(x, p) = H(x, p, u_r(x, p)).$$

Geometric conjugate times

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The variational equation

$$\dot{\delta} z(t) = d\overrightarrow{H_r}(z).\delta z(t)$$

is called the *Jacobi equation* along z. One calls a *Jacobi field* a nontrivial solution $J(t)$ of the Jacobi equation along z. t is said to be *vertical* at time t if

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A time t_c is said to be *geometrically conjugate* if there exists a Jacobi field vertical at 0 and t_c. In which case, $x(t_c)$, is said to be *conjugate* to $x(0)$.

Geometric conjugate times

Denote $exp_t(\overrightarrow{H_r})$ the *flow* of the Hamiltonian vectorfield $\overrightarrow{H_r}$. One defines the *exponential mapping* by

$$exp_{x_{\mathbf{0}},t}(p(0)) \longrightarrow \Pi_x(z(t,z_0)) = x(t,q_0,p_0)$$

where $z(t,z_0)$, with $z(0)=(x(0),p(0))$ is the trajectory of $\overrightarrow{H}$ such that $z(0,z_0)=z_0$.

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Proposition : Let $x_0 \in M$, $L_0 = T^*_{x_0}M$ and $L_t = exp_t(\overrightarrow{H_r})(L_0)$. Then L_t is a Lagrangian submanifold of T^*M whose tangent space is *spanned* by Jacobi fields starting from L_0. Moreover $q(t_c)$ is *geometrically conjugate* to x_0 if and only if exp_{x_0,t_c} *is not a immersion* at p_0.

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⇒ calculating a conjugate point is *equivalent* to verifying a *rank condition*.

Second order optimality condition

Assume the *strong regularity condition* :

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Theorem : Let t_c^1 be the *first* conjugate time along z. The trajectory $q(.)$ is *locally optimal* on $[0, t_c^1)$ in L^∞ topology ; if $t > t_c^1$ then $x(.)$ is *not locally optimal* on $[0, t]$.

Application to space Earth-Moon space transfers

Objective : use optimal control theory to compute *optimal space transfers* in the *Earth-Moon* system

- *time*-minimal space transfers
- *energy*-minimal space transfers

First question : How to *model* the *motion* of a spacecraft in the Earth-Moon system ?

- *Neglect the influences of other planets*
- The spacecraft *does not* affect the motion of the Earth and the Moon
- Eccentricity of orbit of the Moon is *very small* (≈ 0.05)

The Earth-Moon-spacecraft system

The motion of the spacecraft in the Earth-Moon system can be modelled by *the planar restricted 3 body problem*.

Description :

- The Earth (mass M_1) and Moon (mass M_2) are *circularly* revolving aroud their *center of mass* G.
- The spacecraft is *negligeable point mass* M involves in the plane defined by the Earth and the Moon.
- Normalization of the masses : $M_1 + M_2 = 1$
- Normalization of the distance :$d(M_1, M_2) = 1$.

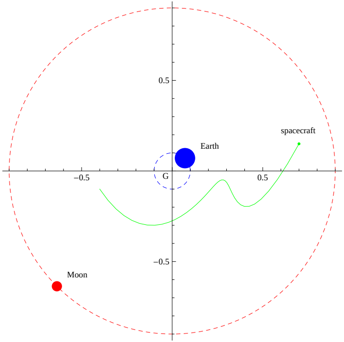

Figure : The circular restricted 3-body problem. The blue dashed line is the orbit of the Earth and the red one is the orbit of the Moon. The trajectory of spacecraft lies in the plan deined by these two orbits.

The Rotating Frame

Idea : Instead of considering a fixed frame $\{G, X, Y\}$, we consider a *dynamic rotating* frame $\{G, x, y\}$ which rotates with the same angular velocity as the Earth and the Moon.

→ *rotation* of angle t

→ *substitution*

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(t)x + \sin(t)y \\ -sin(t)x + \cos(t)y \end{pmatrix}$$

→ *simplifies* the equations of the model

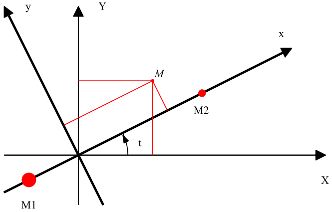

Figure : Comparision between the fixed frame $\{G, X, Y\}$ and the rotating frame $\{G, x, y\}$.

Equations of Motion

In the rotating frame

- define the *mass ratio* $\mu = \frac{M_2}{M_1+M_2}$
- the *Earth* has mass $1-\mu$ and is located at $(-\mu, 0)$;
- the *Moon* has mass μ and is located at $(1-\mu, 0)$;
- *Equations of motion*

$$\begin{cases} \ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y} \end{cases}$$

where

$-V$: is the *mechanical potential*

$$V = \frac{1-\mu}{\varrho_1^3} + \frac{\mu}{\varrho_2^3}$$

ϱ_1 : *distance* between the *spacecraft* and the *Earth*

$$\varrho_1 = \sqrt{(x+\mu)^2 + y^2}$$

ϱ_2 :*distance* between the *spacecraft* and the *Moon*

$$\varrho_2 = \sqrt{(x-1+\mu)^2 + y^2}.$$

Hill Regions

There are 5 *possible* regions of motion, know as the *Hill regions*

Each region is defined by the value of the *total energy* of the system

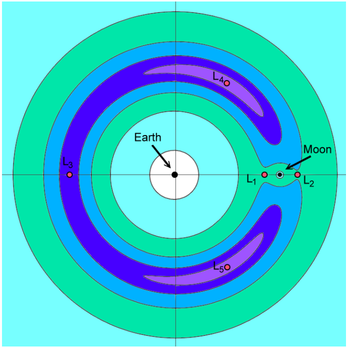

Figure: The Hill regions of the planar restricted 3-body problem

Toplogy/Shape of the regions is determined with respect to the total energy at the *equilibrium points* of the system

Equilibrium points

Critical points of the mechanical potential

→ Points (x, y) where $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$

- *Euler points* : colinear points L_1,L_2, L_3 located on the axis $y = 0$, with

$$x_1 \simeq 1.1557, \quad x_2 \simeq 0.8369, \quad x_1 \simeq -1.0051.$$

- *Lagrange points* : L_4, L_5 which form equilateral triangles with the primaries.

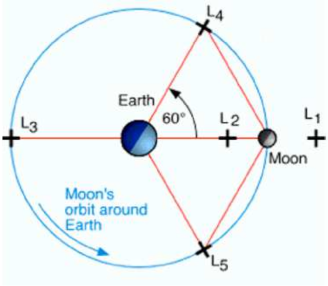

Figure: Equilibrium points of the planar restricted 3-body problem

The controlled restricted 3-Body problem

Control on the motion of the spacecraft ?
→ Thrust/Propulsion provided by the engines of the spacecraft
→ control term $u = (u_1, u_2)$ must be added to the equations of motion
→ *controlled dynamics* of the spacecraft

$$\left\{ \begin{array}{c} \ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x} + u_1 \\ \ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y} + u_2 . \end{array} \right.$$

Setting $q = (x, y, \dot{x}, \dot{y})$
→ *bi-input system*

$$\dot{q} = F_0(q) + F_1(q)u_1 + F_2(q)u_2$$

where

$$F_0(q) = \left(\begin{array}{c} q_3 \\ q_4 \\ 2q_4 + q_1 - (1-\mu)\frac{q_1+\mu}{((q_1+\mu)^2+q_2^2)^{\frac{3}{2}}} - \mu\frac{q_1-1+\mu}{((q_1-1+\mu)^2+q_2^2)^{\frac{3}{2}}} \\ -2q_3 + q_2 - (1-\mu)\frac{q_2}{((q_1+\mu)^2+q_2^2)^{\frac{3}{2}}} - \mu\frac{q_2}{((q_1-1+\mu)^2+q_2^2)^{\frac{3}{2}}} \end{array} \right),$$

$$F_1(q) = \frac{\partial}{\partial q_3}, \ F_2(q) = \frac{\partial}{\partial q_4}$$

Time-minimal problem

Objective : minimizing transfer time between *geostationary* orbit $\mathcal{O}_G$ and a *circular parking* orbit $\mathcal{O}_L$ around the Moon when *low-thrust* is applied.

→ solve the the *optimal control problem*

$$\begin{cases} \dot{q} = F_0(q) + \epsilon(F_1(q)u_1 + F_2(q)u_2), \epsilon > 0 \\ \\ \min_{u(.)\in B_{\mathbb{R}^2}(0,1)} \int_{t_0}^{t_f} dt \\ \\ q(0) \in \mathcal{O}_G, \;\; q(t_f) \in \mathcal{O}_L. \end{cases}$$

where $\epsilon =$ *bound* on the control= *maximum* thrust allowed

Strategy

This optimal control problem problem *can not* be solved analytically

- Highly non-linear
- Singularities

→ *Approximate* low-thrust optimal solutions

- Apply *Pontryagin's Maximum Principle* (necessary conditions)
- Turn the *pseudo*-Hamiltonian system into a *true* Hamiltonian system (Implicit function Theorem)
- Use *shooting method* to compute *extremal curves* of the problem
- Check *local*-optimality of these extremals by using *second order* optimality condition

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  → compute the *first conjugate time* along each extremal and very that it is *greater* than the *transfer time*

Simple shooting method

Idea : Writing boundary and transversality conditions in the form

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⇒ one can use a *Newton* type algorithm.

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4. p_0^N is the zero that we wanted to determine.

Theoretical justification of the continuation method

Theorem :For each λ, the exponential mapping $exp^{\lambda}_{x_{\mathbf{0}},t_f}$ is of textitmaximal rank if and only if the point $x_{\mathbf{1}} = exp^{\lambda}_{x_{\mathbf{0}},t_f}(p(0))$ is *non-conjugate* to $x_{\mathbf{0}}$. Moreover, solutions of the parametrized shooting equation contain a *smooth curve* which can be parametrized by λ and the derivative $E^{'}_{\lambda}$ can be computed integrating the Jacobi equation.

$\Rightarrow$ *convergence* of the smooth continuationis *guaranteed* if there is *no conjugate point* along any extremal curve z_{λ_i}.

Application of the Pontryagin's Maximum Principle

Time-minimal problem $\min_{u \in B_{\mathbb{R}^2}(0,\epsilon)} \int_{t_0}^{t_f} dt$

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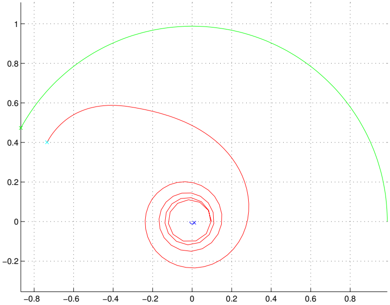
Compute Earth-L_1 trajectories prior to Earth-*Moon* trajectories

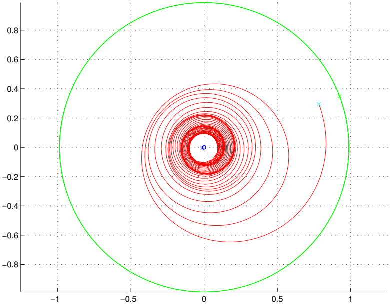
Earth-L_1 time-minimal trajectories



(a) Fixed frame

Figure: Thrust=1N. Red curve : time-minimal transfer to L_1. Green curve : orbit of the Moon.

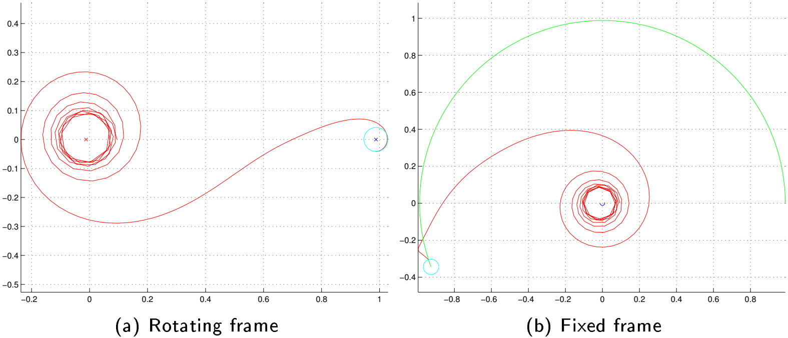
Earth-L_1 time-minimal trajectories



(a) Fixed frame

Figure: Thrust=0.08N. Red curve : time-minimal transfer to L_1. Green curve :orbit of the Moon.

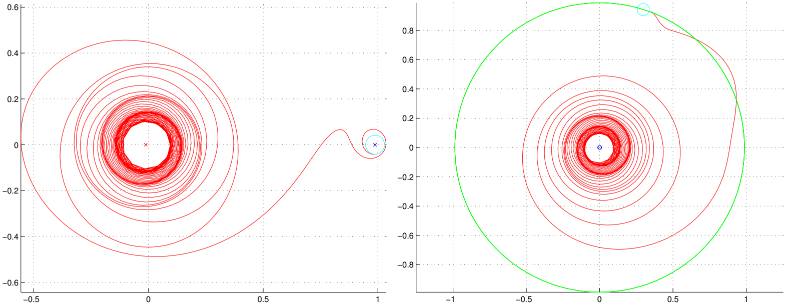
Earth-Moon time-minimal trajectories



(a) Rotating frame (b) Fixed frame

Figure: Thrust=1.Red curve : time-minimal transfer to a circular orbit around the Moon. Green curve :orbit of the Moon. Blue curve : circular parking orbit around the Moon

Earth-Moon time-minimal trajectories



(a) Rotating frame (b) Fixed frame

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Energy-minimal problem

Objective : minimizing *energy-cost* of a transfer between *geostationary* orbit $\mathcal{O}_G$ and a *circular parking* orbit $\mathcal{O}_L$ around the Moon.

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→ solve the the *optimal control problem*

$$\begin{cases} \dot{q} = F_0(q) + F_1(q)u_1 + F_2(q)u_2 \\ \min_{u(.)\in\mathbb{R}^2} \int_{t_0}^{t_f} u_1^2 + u_2^2 \, dt \\ q(0) \in \mathcal{O}_G, \quad q(t_f) \in \mathcal{O}_L. \end{cases}$$

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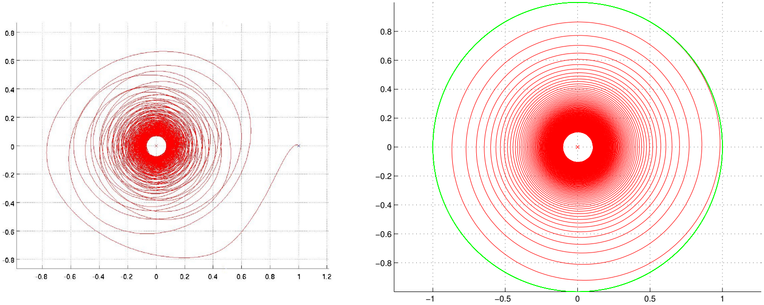
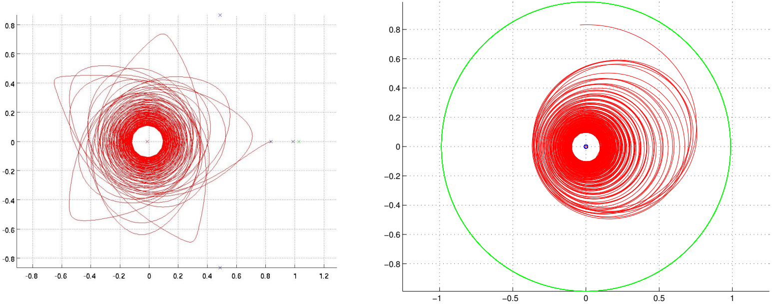
Earth-L_1 energy minimal trajectories



(a) Rotating frame (b) Fixed frame

Figure: μ=0. Red curve : energy-minimal transfer to L_1. Green curve : orbit of the Moon.

Earth-L_1 energy minimal trajectories



(a) Rotating frame (b) Fixed frame

Figure: $\mu = 1.2153e - 2$. Red curve : energy-minimal transfer to L_1. Green curve : orbit of the Moon.

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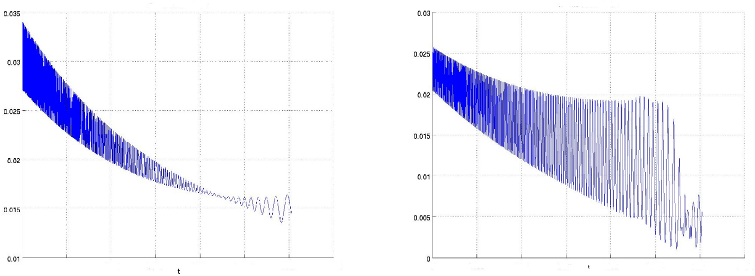
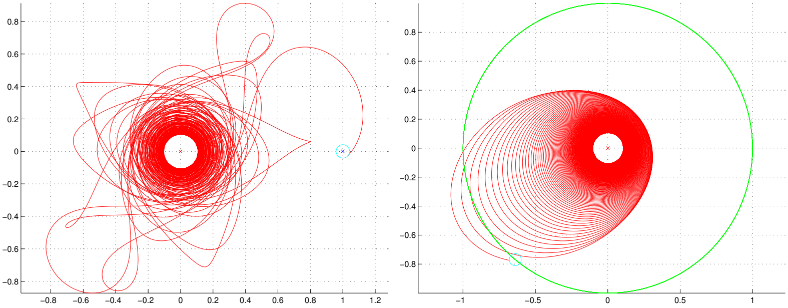


(a) μ=0

(b) $\mu = 1.2153e - 2$

Figure: Control norms

Earth-Moon energy minimal trajectories



(a) Rotating frame

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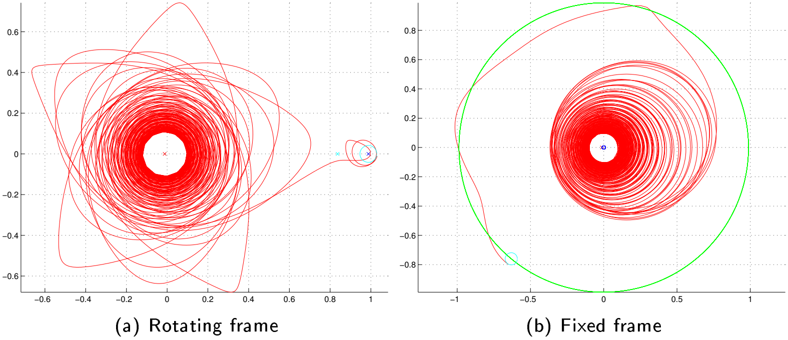

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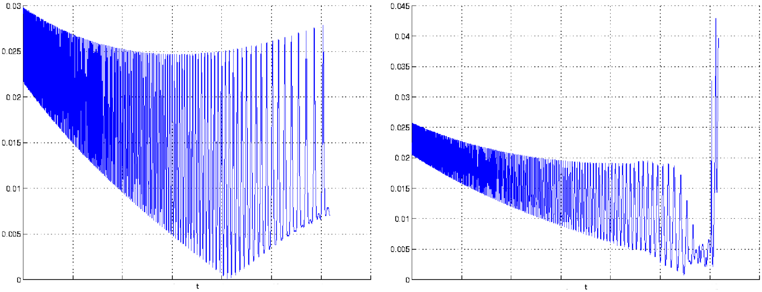

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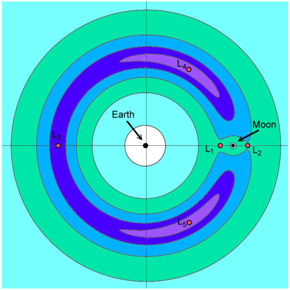
Gravitational model : 3 bodies

Szebehely [1967] *Circular restricted three-body problem*

$$\begin{cases} \ddot{x} = 2\dot{y} - \frac{\partial V}{\partial x} \\ \ddot{y} = -2\dot{x} - \frac{\partial V}{\partial y} \\ \ddot{z} = -\frac{\partial V}{\partial z} \end{cases}$$



where V is the potential energy function :

$$-V = \frac{x^2 + y^2}{2} + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{\mu(1-\mu)}{2}$$

and

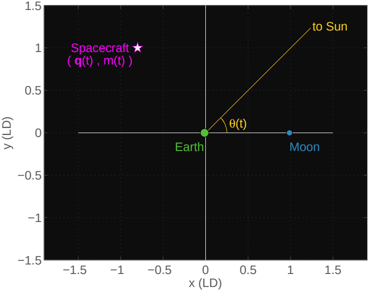
$$\rho_1 = \sqrt{(x+\mu)^2 + y^2 + z^2} \qquad \rho_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Gravitational model : 4 bodies

Simó *et al.* [1995] *Circular restricted four-body problem*

Sun at $(r_s \cos\theta, r_s \sin\theta)$ with mass μ_s

$$\begin{cases} \ddot{x} = 2\dot{y} - \frac{\partial V_4}{\partial x} \\ \ddot{y} = -2\dot{x} - \frac{\partial V_4}{\partial y} \\ \ddot{z} = -\frac{\partial V_4}{\partial z} \end{cases}$$



where

$$V_4 = V - V_s, \qquad V_s(t) = \frac{\mu_s}{\rho_s} - \frac{\mu_s}{r_s^2}(x\cos\theta + y\sin\theta)$$

$$\rho_s = \sqrt{(x - r_s\cos\theta)^2 + (y - r_s\sin\theta)^2 + z^2}, \qquad \mu_s = 329012.5, \qquad r_s = 389.2,$$

$$\dot{\theta} = \omega_s = -0.925 \qquad \rightarrow \qquad \theta(t) = \omega_s t + \theta_0$$

Notation and drift equations

State variables :

$$\mathbf{X}(t) = \begin{pmatrix} \mathbf{q}(t) \\ m(t) \end{pmatrix} = \begin{pmatrix} \mathbf{s}(t) \\ \mathbf{v}(t) \\ m(t) \end{pmatrix}$$

Uncontrolled system :

$$\dot{\mathbf{X}} = \mathbf{F}_0(\mathbf{X}) = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 2\dot{y} + x - (1-\mu)\frac{x+\mu}{\rho_1^3} - \mu\frac{x-1+\mu}{\rho_2^3} - \mu_s\frac{x - r_s\cos\theta}{\rho_s^3} - \frac{\mu_s}{r_s^2}\cos\theta \\ -2\dot{x} + y - (1-\mu)\frac{y}{\rho_1^3} - \mu\frac{y}{\rho_2^3} - \mu_s\frac{y - r_s\sin\theta}{\rho_s^3} - \frac{\mu_s}{r_s^2}\sin\theta \\ -(1-\mu)\frac{z}{\rho_1^3} - \mu\frac{z}{\rho_2^3} - \mu_s\frac{z}{\rho_s^3} \\ 0 \end{pmatrix}$$

Control system

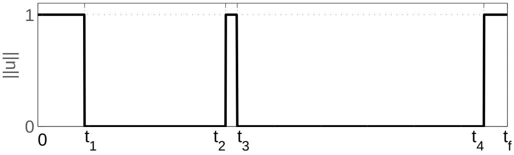
Control variables :

$$\mathbf{u}(t) = \left(\begin{array}{c} u_1(t) \\ u_2(t) \\ u_3(t) \end{array} \right), \qquad \|\mathbf{u}\| \leq 1, \qquad \mathbf{u} \in \overline{\mathcal{U}}$$

Set of admissible controls :

$$\overline{\mathcal{U}} = \left\{ \mathbf{u} : \mathbb{R} \to B_{\mathbb{R}^3}(0,1) : \begin{array}{c} \mathbf{u} \text{ measurable, } \|\mathbf{u}(t)\| = \bar{\alpha}(t) \text{ for a sequence} \\ \text{of times } 0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_f \end{array} \right\}$$

$$\bar{\alpha}(t) = \left\{ \begin{array}{ll} 1, & t \in [0, t_1] \cup [t_2, t_3] \cup [t_4, t_f] \\ 0, & \text{otherwise} \end{array} \right\}$$



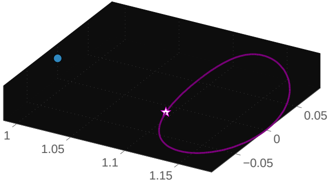
$t_i, i = 1..4$ are called *switching times*

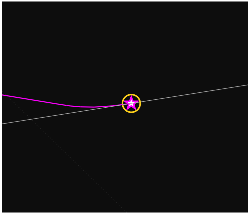
Minimal fuel problem

We can now state the minimal cost problem :

$$(\star)_D \begin{cases} \min_{\mathbf{u}\in\overline{\mathcal{U}}} \int_0^{t_f} \|\mathbf{u}(t)\| dt \\ \dot{\mathbf{X}} = \mathbf{F}_0(\mathbf{X}) + \frac{T_{max}}{m} \sum_{i=1}^{3} u_i \mathbf{F}_i - T_{max}\beta\|\mathbf{u}\|\mathbf{F}_4 \\ \mathbf{q}(0) = \mathbf{q}_{dprt}, \quad m(0) = m_0 \\ \mathbf{q}(t_f) = \mathbf{q}_{rdvz}, \quad \theta(t_f) = \theta_{rdvz} \end{cases}$$

D is the *problem data* $D = (t_f, \mathbf{q}_{dprt}, \mathbf{q}_{rdvz}, \theta_{rdvz})$





Pontryagin maximum principle

(Pontryagin [1962]) Let $(\mathbf{X}, \mathbf{u})$ be an optimal pair for $(\star)_D$.
Then there exists an absolutely continuous function $\mathbf{P}(\cdot)$, called the costate vector, and a constant $p_0 \leq 0$, $(\mathbf{P}, p_0) \neq (\mathbf{0}, 0)$, such that for a.e. t we have :

Pseudo-Hamiltonian Equations :

$$\dot{\mathbf{P}} = -\frac{\partial H}{\partial \mathbf{X}}, \quad \dot{\mathbf{X}} = \frac{\partial H}{\partial \mathbf{P}}$$

where $H(t, \mathbf{P}, \mathbf{X}, \mathbf{u}) = p_0 \|\mathbf{u}\| + \langle \mathbf{P}, \dot{\mathbf{X}} \rangle$.

Maximization Condition :

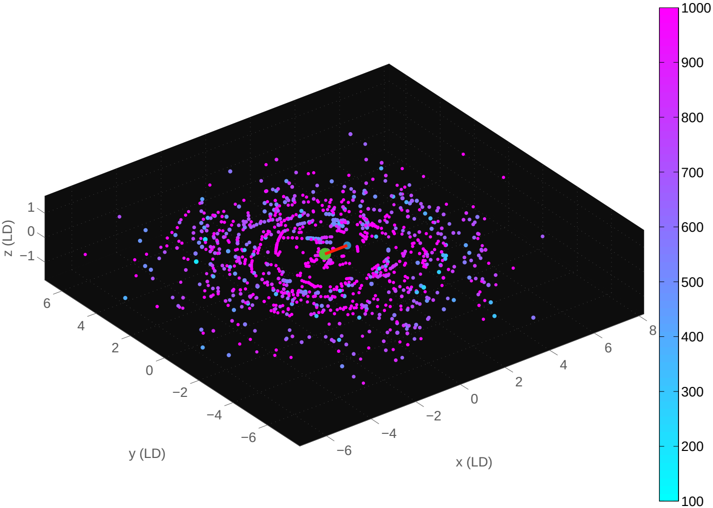
$$H(t, \mathbf{P}(t), \mathbf{X}(t), \mathbf{u}(t)) = \max_{\mathbf{v} \in B_{\mathbb{R}^3}(0,1)} H(t, \mathbf{P}(t), \mathbf{X}(t), \mathbf{v})$$

Transversality Condition :

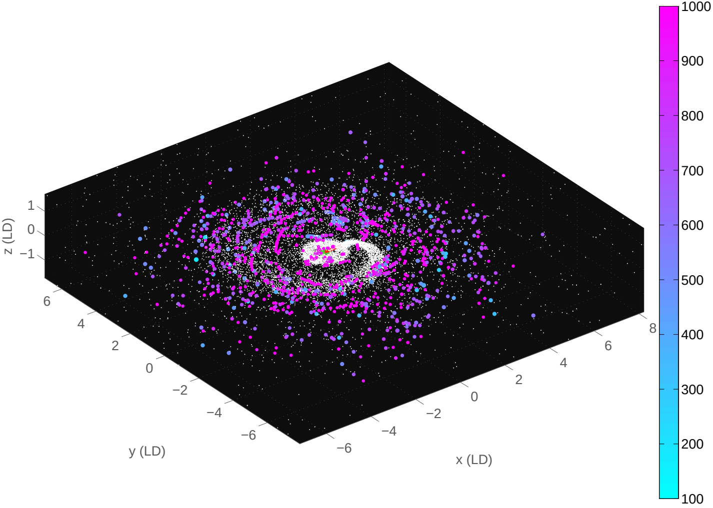
$$\mathbf{P}(0) \perp T_{\mathbf{X}(0)} M_0, \qquad \mathbf{P}(t_f) \perp T_{\mathbf{X}(t_f)} M_f$$

A solution $(\mathbf{P}, \mathbf{X}, \mathbf{u})$ is called an *extremal*.

ΔV at rendezvous (m/s)



ΔV at continuation points (m/s)

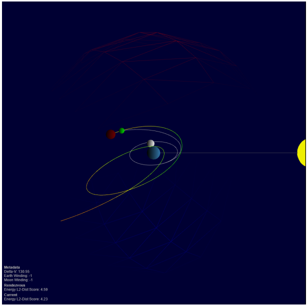


Visualization research

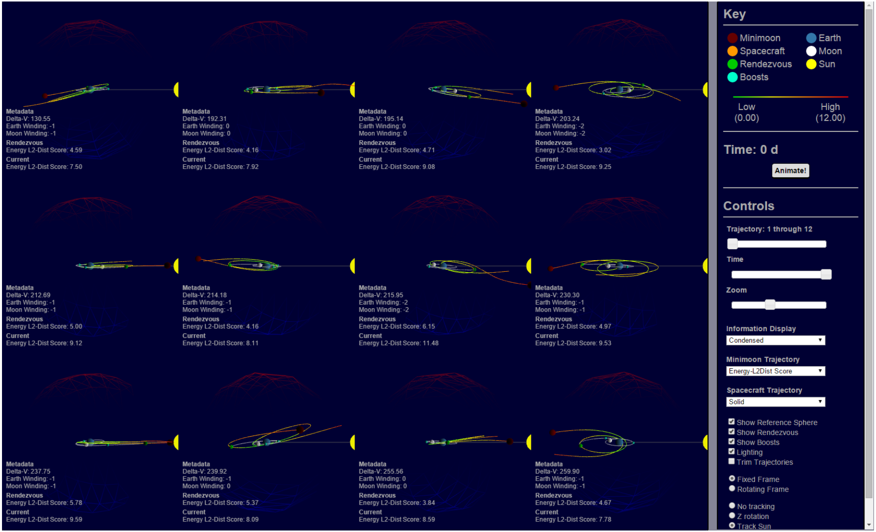
Further questions :

- Why were some transfers better than others ?
- What characterizes a good candidate asteroid ?
- What some general features that allow low cost transfers ?

This prompted exploratory work using visualization and statistical analysis.



Visualization demo



Metadata
Delta-V: 130.55
Earth Winding: -1
Moon Winding: -1
Rendezvous
Energy L2-Dist Score: 4.59
Current
Energy L2-Dist Score: 7.50

Metadata
Delta-V: 192.31
Earth Winding: 0
Moon Winding: 0
Rendezvous
Energy L2-Dist Score: 4.16
Current
Energy L2-Dist Score: 7.92

Metadata
Delta-V: 195.14
Earth Winding: 0
Moon Winding: 0
Rendezvous
Energy L2-Dist Score: 4.71
Current
Energy L2-Dist Score: 9.08

Metadata
Delta-V: 203.24
Earth Winding: -2
Moon Winding: -2
Rendezvous
Energy L2-Dist Score: 3.02
Current
Energy L2-Dist Score: 9.25

Metadata
Delta-V: 212.69
Earth Winding: -1
Moon Winding: -1
Rendezvous
Energy L2-Dist Score: 5.00
Current
Energy L2-Dist Score: 9.12

Metadata
Delta-V: 214.18
Earth Winding: -1
Moon Winding: -1
Rendezvous
Energy L2-Dist Score: 4.16
Current
Energy L2-Dist Score: 8.11

Metadata
Delta-V: 215.95
Earth Winding: -2
Moon Winding: -2
Rendezvous
Energy L2-Dist Score: 6.15
Current
Energy L2-Dist Score: 11.48

Metadata
Delta-V: 230.30
Earth Winding: -1
Moon Winding: -1
Rendezvous
Energy L2-Dist Score: 4.97
Current
Energy L2-Dist Score: 9.53

Metadata
Delta-V: 237.75
Earth Winding: -1
Moon Winding: -1
Rendezvous
Energy L2-Dist Score: 5.78
Current
Energy L2-Dist Score: 9.59

Metadata
Delta-V: 239.92
Earth Winding: -1
Moon Winding: 0
Rendezvous
Energy L2-Dist Score: 5.37
Current
Energy L2-Dist Score: 8.09

Metadata
Delta-V: 255.56
Earth Winding: 0
Moon Winding: 0
Rendezvous
Energy L2-Dist Score: 3.84
Current
Energy L2-Dist Score: 8.59

Metadata
Delta-V: 259.90
Earth Winding: -1
Moon Winding: -1
Rendezvous
Energy L2-Dist Score: 4.67
Current
Energy L2-Dist Score: 7.78

Time: 0 d

Trajectory: 1 through 12
Time
Zoom
Information Display: Condensed
Minimoon Trajectory: Energy-L2Dist Score
Spacecraft Trajectory: Solid

☑ Show Reference Sphere
☑ Show Rendezvous
☑ Show Boosts
☑ Lighting
☐ Trim Trajectories

○ Fixed Frame
○ Rotating Frame

○ No tracking
○ Z rotation
○ Track Sun

Observations and hypotheses

Potential predictors based on visualization and talks with industry :

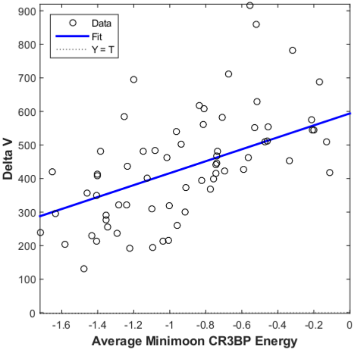
- Average energy
- Average velocity
- Planarity
- Lunar planarity
- Z-displacement
- Distance from barycenter
- Eccentricity
- Average curvature

Statistical results

The most important factors were :

- Average energy
- Lunar planarity
- Variance of distance from barycenter

[include other graphs here]



Conclusions

The following are fair predictors of good transfers :

- Low average energy relative to the departure energy (from earth-moon L_2)
- Minimoon trajectories that lie mostly in the lunar plane
- Roughly circular geocentric orbits

For mission design, this implies that parking an array of spacecraft at various energy levels could be a viable solution to maximize low cost interceptions. Furthermore, prioritizing minimoons with roughly moon-like orbits could help reduce fuel costs. This has implications for missions like NASA's ARM as well.